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Some Propertiesofthe Latticeof Convex Edge Setsofa Connected Directed Graph

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ABSTRACT

Let G be a connected directed graph and E(G) be the directed edge set of G. A subset C of E(G) is said to be convex if for any $e_i, e_j \in C$, there is a directed path containing e_i, e_j and the edge set of every $e_i - e_j$ geodesic is contained in C. Let Con(G) be the set of all convex edge sets of G together with empty set partial ordered by set inclusion relation. Then Con(G) forms a lattice if and only if G has an Euler trial. In this paper cardinality of the lattice Con(G) is discussed. Also some of the properties of the lattice Con(G) are studied.

Indexterm: Lattices,Chains, Irreducibility, Connected digraphs, Convex edge sets, Paths, Cycles

MSC: 06B99, 05C20, 05C38

1. Introduction

Motivated by the studies on the lattice of convex sets of a connected graph [8], the set of convex edge sets of connected digraphs together with empty set is considered in [1] and it is found that this set forms a lattice with respect to the partial order set inclusion if and only if digraph contains an Euler trail. In this paper we studied properties of these lattices when the digraph G is directed path and directed cycle. Also irreducibility criteria and conditions under which con(G) becomes lower semimodularis discussed. It is proved that if |E(G)| ≥ 3, Con(G) satisfies lower covering condition if and only if G is a directed cycle C_3.

For terminologies and notations used in this paper we refer to [3] and [4]

2. Preliminaries

Let G be a finite connected digraph. E(G) be the edge set of G. A set C ⊆ E (G) is said to be convex in G if for every two edges e_i, e_j∈ C, there is a directed path containing e_i, $e_j$and the edge set of every e_i- $e_j$geodesic (i.e shortest directed path containing e_i and e_j) is contained in C. In a digraph G, a walk in which no edge is repeated is a (directed) trail. A closed walk in which no edge is repeated is a (directed) circuit. A trail containing all the edges of G is Euler trail and a circuit containing all the edges of G is Euler circuit.An element 'a' of a lattice L is join irreducible if $a = b \vee c$ implies that $a = b$or $a = c$. 'a' is meet irreducible if $a = b \wedge c$ implies that $a = b$or $a = c$. An element which is both meet and join irreducible is called doubly irreducible. A lattice L is said to satisfy the lower covering conditionif for $a, b \in L$ $a \wedge b \prec b$ implies $a \prec a \vee b$. A lattice L is lower semimodular(LSM) if $a \vee b$covers both a and b implies that both a and b cover $a \wedge b$.

For a finite connected digraph G, let the set of all convex edge sets in G together with empty set be denoted by Con(G). Define a binary relation ≤ on Con(G) by, for A,B ∈ Con(G), A ≤ B if and only if A ⊆ B. Then clearly ≤ is a partial order on Con(G). Moreover < Con(G), ⊆> forms a lattice where for A,B ∈Con(G), A ∧ B = A∩B and A ∨ B =<A∪B> is the smallest convex edge set containing A∪B.

For example, the lattice given in Fig 2.2 represents the lattice <Con(G), ⊆>of the connected digraph G given in Fig 2.1.

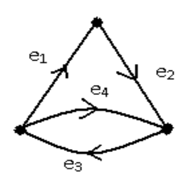

Fig 2.1

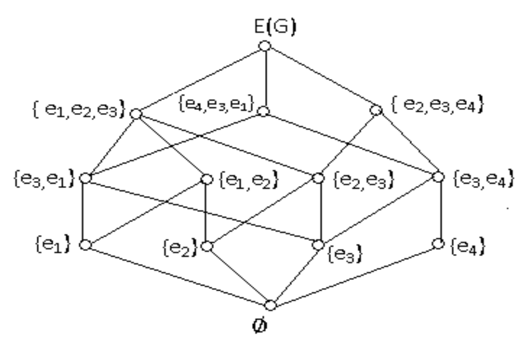

Fig 2.2

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Hereafter we consider digraph G containing an Euler trail and use Con(G) to represent the lattice< Con(G), ⊆>

3. On the Lattice Con(G)

Remark 3.1:*Con(G) is a chain if and only if G is a directed graph with single edge.*

Remark 3.2: *If G is a directed graph with two edges, then Con(G) will be as shown in Fig 3.1 which is a Boolean algebra.*

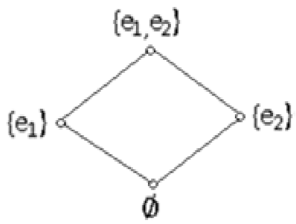

Fig 3.1

Theorem 3.3:*If G is a directed cycle with n edges, then* $|Con(G)| = \left(\left\lceil\frac{n}{2}\right\rceil \times n\right) + 2$

$$\left(Where \left\lceil\frac{n}{2}\right\rceil = smallest\ integer \geq \frac{n}{2}\right)$$

Proof: Let G be the directed cycle. There are n convex sets with single element, n convex sets with twoelements, and so on, finally n convex sets with $\left\lceil\frac{n}{2}\right\rceil$ elements. Hence there are $\left\lceil\frac{n}{2}\right\rceil \times n$ such convex sets. Therefore |Con(G)|= $\left(\left\lceil\frac{n}{2}\right\rceil \times n\right) + 2$, including ∅ and E(G).

Theorem 3.4: If *G is a directed path with n edges, then* $|Con(G)| = \frac{n(n+1)}{2} + 1$

Proof: There are n convex sets with single edge, $n-1$ convex sets with two edges, $n-2$ convex sets with threeedges and so on, finally one convex set with n edges. Including empty set,

$$|\text{Con(G)}| = n + (n-1) + (n-2) + \cdots + 1 + 1 = \frac{n(n+1)}{2} + 1$$

Theorem 3.5:*An element A ∈Con(G) is doubly irreducible if and only if* $A = \{e_i\}$ *where* $e_i = \{u, v\}$ *is a pendant edge with indegree of u is 0 or 1 and outdegree of v is 1 or 0 respectively OR (u,v) and (v,u) is a directed cycle with indegree of u=1 and outdegree of v=1.*

Proof: Let A ∈Con (G) be doubly irreducible.If A contains more than one element Say$A = \{e_1, e_2, \dots e_n\}$, then

$A = \bigvee_{i=1}^{n}\{e_i\}$and therefore $A = \{e_i\}$ for some i, since A is join irreducible.

If $e_i = \{u, v\}$ is not a pendant edge, then indegree of u is one or more and outdegreeof v is one or more, then there will be a directed path $e_j e_i e_k$. Then $\{e_i\} = \{e_j, e_i\} \wedge \{e_i, e_k\}$, contradiction to $\{e_i\}$ is meet irreducible.Let $e_i = \{u, v\}$ be a pendant edge with indegree of u = 2 or more, then there will be edges e_j and e_k such that $e_j = \{u_1, u\}$, $e_k = \{u_2, u\}$ and $\{e_i\} = \{e_j, e_i\} \wedge \{e_k, e_i\}$, contradiction to $\{e_i\}$ is meet irreducible. Similarly if $e_i = \{u, v\}$ is a pendant edge with outdegree of v = 2 or more, then we get a contradiction to$\{e_i\}$ is meet irreducible. Thus if indegree of u=2 or more OR outdegree of v=2 or more, then $\{e_i\}$ becomes meet reducible.

Conversely $A = \{e_i\}$ is join irreducible. If A is meet reducible say $A = B \wedge C = B \cap C$ for some $B, C \in Con(G)$ such that $A \neq B, A \neq C$ .Then $\{e_i\} \in B \cap C$. Consider $\{e_j\} \in B$, $\{e_k\} \in C$ where $e_j \neq e_i$, $e_k \neq e_i$. Let $e_i f_1 f_2 \dots e_j$ be the shortest path connecting e_i, e_j in B. Also let $e_i g_1 g_2 \dots e_k$ be the shortest path connecting e_i, e_k in C.If $f_1 = g_1$,then $f_1 \in B \cap C$ contrdiction to $B \cap C = \{e_i\}$ .Also if $f_1 \neq g_1$,then outdegree of $v > 1$ contradiction to the fact that outdegree of v is atmost 1.

Theorem 3.6:*Let G be a directed graph with |E(G)| ≥ 3.Con(G) satisfies lower covering condition if and only if*

G is a directed cycleC_3.

Proof: If G is C_3 ,then Con(G) is distributive[1] and hence Con(G) satisfies lower covering condition.

Conversely, letCon(G) satisfies lower covering condition. If G is not C_3 ,then G contains a trail(which is not a circuit) say $e_i e_j e_k$. Clearly$\emptyset = \{e_i\} \wedge \{e_k\} \prec \{e_i\}$.But $\{e_k\} \prec \{e_j, e_k\} < \{e_i\} \vee \{e_k\}$. Which implies $\{e_k\} \nprec \{e_i\} \vee \{e_k\}$ contradiction to Con(G) satisfies lower covering condition.Hence G must be C_3.

Theorem 3.7:*Con(G) is lower semimodular(LSM) in the following cases.*

1) *G is a directed cycle* C_3
2) *G is of the form given in Fig 2.1*
3) *G is a directed path or directed path containing two element cycles at its end vertices.*
4) *G is a directed path containing three element cycles at its end vertices.*

Proof: If G is a directed cycle C_3, then Con(G) is modular[1]. Every modular lattice is LSM.

If G is of the form given in Fig 2.1, then Con(G) will be as shown in Fig 2.2. Clearly it is LSM.

Let G be as given in case 3. Let $e_1 e_2 \dots e_n$ be the Euler Trail.All possible convex edge sets are as follows.

Emptyset,

$\{e_1\}$,$\{e_2\}$…$\{e_n\}$,$\{e_1, e_2\}$,$\{e_2, e_3\}$…$\{e_{n-1}, e_n\}$,$\{e_1, e_2, e_3\}$,$\{e_2, e_3, e_4\}$... $\{e_{n-2}, e_{n-1}, e_n\}$ Continuing like this $\{e_1, e_2, \dots e_n\}$ is the maximum element. If $A \vee B$ covers both A and B , then $A, B, A \vee B$ are of the form $\{e_i, e_{i+1}, \dots e_k\}$, $\{e_{i+1}, e_{i+2} \dots e_{k+1}\}$ and $\{e_i, e_{i+1}, \dots e_{k+1}\}$ respectively. Clearly $A \wedge B$ =

$\{e_{i+1}, e_{i+2} \dots e_k\}$ and it is covered by both A and B. Thus Con(G) is LSM.
Similarly, if G is a directed path containing 3 element cycles at its end vertices, then Con(G) will be LSM.

Theorem 3.8: *If G contains a cycle of length ≥ 4, then Con(G) is not LSM.*

Proof: Let G contains a directed cycle C_n. Say $e_1 e_2 \dots e_n$ with n ≥ 4.
Case 1: When n is even
Taking $A = \left\{e_1, e_2, \dots e_{\left\lceil\frac{n}{2}\right\rceil}\right\}$ and $B = \left\{e_{\left\lceil\frac{n}{2}\right\rceil+1}, e_{\left\lceil\frac{n}{2}\right\rceil+2}, \dots, e_n\right\}$, we get $A \vee B = \{e_1, e_2, \dots e_n\}$ and $A \wedge B = \emptyset$. Clearly $A \vee B$ covers both A and B. But $A \wedge B$ is not covered by A and B. Because $\emptyset$ is always covered by singleton sets. Here both A and B contain two or more elements (since n ≥ 4).
Case 2: When n is odd
Taking $A = \left\{e_1, e_2, \dots e_{\left\lceil\frac{n}{2}\right\rceil}\right\}$ and $B = \left\{e_{\left\lceil\frac{n}{2}\right\rceil}, e_{\left\lceil\frac{n}{2}\right\rceil+1}, \dots, e_n\right\}$, we get $A \vee B = \{e_1, e_2, \dots e_n\}$ and $A \wedge B = \left\{e_{\left\lceil\frac{n}{2}\right\rceil}\right\}$. Clearly $A \vee B$ covers both A and B. But $A \wedge B$ is not covered by A and B. Because singleton sets are always covered by two element sets. Here both A and B contain three or more elements (since n ≥ 5).

Remark 3.9: *If G is any of the forms given in Theorem 3.7, then Con(G) is LSM and hence they satisfy Jordan Dedekind chain condition. Infact if G is a directed cycle C_n, then it can be observed that Con(G) satisfies Jordan Dedekind chain condition. As there are n convex sets with single element, n convex sets with two elementsand so on, finally n convex sets with $\left\lceil\frac{n}{2}\right\rceil$ elements. Empty set is covered by single element set. Single element sets are covered by two element sets. Two element sets are covered by three element sets continuing like this, $\left\lceil\frac{n}{2}\right\rceil$ element sets are covered by $E(G)$. Therefore all maximal chains connectingany two elements of Con(G) are of the same length.*

References:

[1] Asha Saraswathi B. and Lavanya S.,On the Lattice of Convex Edge sets of a Connected directed graph.,International Journal of Pure and Applied Mathematical Sciences.Volume 8, Number 2(2015),pp.155-161
[2] Birkhoff. G. Lattice theory, Third edition (New York, 1967)
[3] Gratzer, G: General lattice theory, BirkhauserVerlag, academic press, 1978
[4] Harary F: Graph theory, Addision-wesley 1969
[5] D. Kelly and I. Rival, Planar lattices, CanadJ. Math, 27 No. 3, (1975), 636-665
[6] Koh. K. M. On the sublattices of a lattice, Nanta Math. 6(1)(1973), 68-79.
[7] Lavanya. S and S. Parameswara Bhatta, A new approach to the lattice of convex Sublattices of a lattice, Algebra universalis, 35(1996), 63-71
[8] Lavanya S. and Subramanya Bhat S., On the lattice of convex sets of a connected graph, Global journal of pure and applied Mathematics, Vol 7, 2(2011), 157-162
[9] Lavanya S., On the lattice of path sets of a connected graph, Indian Journal of Mathematics research, Vol. 1, No.2 (2013), 219-222.
[10] Pfaltz. J. L., Convexity in directed graphs, J. Combinatorial theory 10(1971), 143-162.